Models of Set Theory I – Summer 2017

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Problem 21 [4 points]
Let $\mathbb{P} = (P, \leq, 1)$ be a partial order. We call an antichain A of $\mathbb{P}$ *maximal* in case every $p \in P$ is compatible to some $a \in A$. We say that a set $D \subseteq P$ is predense in $\mathbb{P}$ in case every $p \in P$ is compatible to some $d \in D$ (so - trivially - an antichain of $\mathbb{P}$ is maximal if and only if it is predense). Verify the following:

- If $D \subseteq P$ is dense, then there is $A \subseteq D$ such that A is a maximal antichain of $\mathbb{P}$.
- If $A \subseteq P$ is a maximal antichain of $\mathbb{P}$, then $\{p \in P \mid \exists a \in A \; p \leq a\}$ is a dense subset of $\mathbb{P}$.
- Find an example of a partial order $\mathbb{P}$, and a predense subset D of P, such that no subset of D is a maximal antichain of $\mathbb{P}$. (*Hint:* There is a suitable partial order the domain of which has 6 elements.)

Problem 22 [3 points] Show that the following are equivalent for a partial order $\mathbb{P}$, a countable ground model M (that is M is transitive and satisfies ZFC), and a filter G on $\mathbb{P}$.

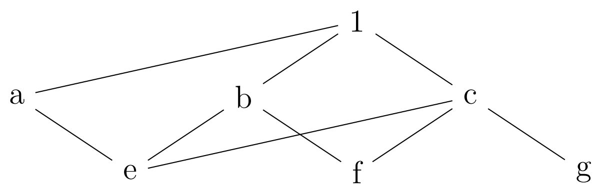
- $G \cap D \neq \emptyset$ for every dense subset D of P in M.
- $G \cap A \neq \emptyset$ for every maximal antichain A of P in M.
- $G \cap D \neq \emptyset$ for every predense $D \subseteq P$ in M.

Problem 23 [6 points] Fix a countable ground model M and a partial order $\mathbb{P} \in M$. Verify the following.

- (Maximality Principle) Show that if $p \Vdash \exists x \, \varphi(x)$ for some first order formula φ in the language of set theory, then there is a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$.
- We say that $p \Vdash \dot{x} \in M$ if $\dot{x}^G \in M$ whenever G is $\mathbb{P}$-generic over M. Show that if $p \Vdash \dot{x} \in M$, then there is $q \leq p$ and a set $y \in M$ such that $q \Vdash \dot{x} = \check{y}$.
- Show that the following are equivalent:
  - $p \Vdash \dot{x} \in M$.
  - $\forall q \leq p \, \exists r \leq q \; \exists y \in M \; r \Vdash \dot{x} = \check{y}$.
  - $\exists B \in M \; p \Vdash \dot{x} \in \check{B}$.

Problem 24 [7 points]

Let $P = \{1, a, b, c, e, f, g\}$ and let $\leq = \{(e,a), (e,b), (e,c), (f,b), (f,c), (g,c), (g,d)\} \cup \{(x,1) \mid x \in P\} \cup \{(x,x) \mid x \in P\}$. We illustrate the ordering of $\mathbb{P} = (P, \leq, 1)$ in the picture below.



- Calculate the separative quotient $\mathbb{P}/\!\sim$ of $\mathbb{P}$, as defined on Problem Sheet 5.
- Use $\mathbb{P}/\!\sim$ to show that is generally not the case that $[x]_\sim \leq [y]_\sim$ implies that $\exists x^* \in [x]_\sim \, \exists y^* \in [y]_\sim \; x^* \leq y^*$.
- Show that if $\mathbb{P} = (P, \leq_\mathbb{P}, 1_\mathbb{P})$ is any separative partial order, then there is a Boolean algebra $\mathbb{B}$ with the ordering $\leq_\mathbb{B}$ as defined on Problem Sheet 5, such that P *is a dense subset of* B^*, that is:
  - $P \subseteq B$, $\leq_\mathbb{P} \subseteq \leq_\mathbb{B}$, $1_\mathbb{P} = 1_\mathbb{B}$, and
  - $\forall b \in B^* \, \exists p \in P \; p \leq_\mathbb{B} b$.

Hint: Construct $\mathbb{B}$ in ω-many steps.

- Show that if $\mathbb{P} = (P, \leq_\mathbb{P}, 1_\mathbb{P})$ and $\mathbb{Q} = (Q, \leq_\mathbb{Q}, 1_\mathbb{Q})$, Q is a dense subset of P, and both $\mathbb{P}$ and $\mathbb{Q}$ are elements of some countable ground model M, then the $\mathbb{P}$-generic extensions of M are exactly the $\mathbb{Q}$-generic extensions of M. Thus this holds true in particular if $\mathbb{Q} = \mathbb{B}$ is the Boolean algebra defined from $\mathbb{P}$ above.
- Show that whenever $\mathbb{B}$ is a Boolean algebra, then there exists an ultrafilter U on $\mathbb{B}$, that is U is a filter on $\mathbb{B}$ such that either p or $\neg p$ is an element of U for every $p \in \mathbb{B}$.
- Show that if $\mathbb{P} \in M$ is a forcing notion which is non-atomic (i.e. $\forall p \in P \, \exists q, r \leq p \; q \perp r$), M is a countable ground model, and G is an M-generic filter on $\mathbb{P}$, then $G \notin M$. Infer that ultrafilters for Boolean algebras in M are not necessarily M-generic.

Hint: Assume for a contradiction that $G \in M$, and show that then $P \setminus G \in M$ is a dense subset of P. Use this to obtain a contradiction.